Transformations of polynomial ensembles

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Dedicated to Ed Saff on the occasion of his 70th birthday

Abstract. A polynomial ensemble is a probability density function for the position of n real particles of the form $\frac{1}{Z_n}\prod_{j<k}(x_k-x_j)\,\det\left[f_k(x_j)\right]_{j,k=1}^n$, for certain functions $f_1,\ldots,f_n$. Such ensembles appear frequently as the joint eigenvalue density of random matrices. We present a number of transformations that preserve the structure of a polynomial ensemble. These transformations include the restriction of a Hermitian matrix by removing one row and one column, a rank-one modification of a Hermitian matrix, and the extension of a Hermitian matrix by adding an extra row and column with complex Gaussians.

1. Polynomial ensembles

A polynomial ensemble is a probability density function on $\mathbb{R}^n$ of the form

$$\mathcal{P}(x_1,\ldots,x_n)=\frac{1}{Z_n}\,\Delta_n(x)\,\det\left[f_k(x_j)\right]_{j,k=1}^n, \tag{1.1}$$

where $f_1,\ldots,f_n$ is a given sequence of real-valued functions,

$$\Delta_n(x)=\prod_{j<k}(x_k-x_j)=\det[x_j^{k-1}]_{j,k=1}^n,\qquad x=(x_1,\ldots,x_n)\in\mathbb{R}^n,$$

denotes the Vandermonde determinant, and Z_n is a normalization constant. Certain conditions on the functions $f_1,\ldots,f_n$ have to be satisfied to ensure that (1.1) is indeed a probability density. For example, the functions should be linearly independent and the integrals

$$\int_{-\infty}^{\infty}x^{j-1}f_k(x)dx,\qquad j,k=1,\ldots,n$$

should be convergent. In addition, (1.1) has to be non-negative for all possible choices of $(x_1,\ldots,x_n)\in\mathbb{R}^n$. The probability density (1.1) only depends on the linear span of the functions $f_1,\ldots,f_n$, as one can see by applying column transformations to the second determinant in (1.1).

A special case arises if $f_k(x)=x^{k-1}w(x)$, for $k=1,\ldots,n$, with w an integrable non-negative function on $\mathbb{R}$ such that the moments up to order $2n-2$ exist. In that case (1.1) can be written as

$$\mathcal{P}(x_1,\dots,x_n) = \frac{1}{Z_n}\,\Delta_n^2(x)\prod_{j=1}^n w(x_j), \qquad x=(x_1,\dots,x_n)\in\mathbb{R}^n, \tag{1.2}$$

and (1.2) is known as an orthogonal polynomial ensemble [**19**], as the analysis of (1.2) relies on the polynomials that are orthogonal with respect to w. The ensembles (1.2) arise as the joint probability density of eigenvalues of unitary invariant ensembles of Hermitian random matrices [**10**]. The polynomial ensembles (1.1) also include the multiple orthogonal polynomials ensembles, see [**20**], where also more examples from random matrix theory are given.

On the other hand, we have that (1.1) is a special case of the more general class of biorthogonal ensembles, see [**8, 9**],

$$\mathcal{P}(x_1,\dots,x_n) = \frac{1}{Z_n}\,\det\left[g_k(x_j)\right]_{j,k=1}^n \det\left[f_k(x_j)\right]_{j,k=1}^n, \tag{1.3}$$

which involves two sequences $g_1,\dots,g_n$ and $f_1,\dots,f_n$ of given functions. It is known that (1.3) is determinantal [**8**], which means that there exists a kernel $K_n : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

$$\mathcal{P}(x_1,\dots,x_n) = \frac{1}{n!}\det\left[K_n(x_j,x_k)\right]_{j,k=1}^n$$

and such that for every $m=1,\dots,n-1$,

$$\int_{\mathbb{R}^{n-m}} \mathcal{P}(x_1,\dots,x_n)dx_{m+1}\cdots dx_n = \frac{(n-m)!}{n!}\det\left[K_n(x_j,x_k)\right]_{j,k=1}^m.$$

The correlation kernel K_n has the form

$$K_n(x,y) = \sum_{k=1}^n \psi_k(x)\phi_k(y)$$

where $\operatorname{span}\{\psi_1,\dots,\psi_n\} = \operatorname{span}\{g_1,\dots,g_n\}$, $\operatorname{span}\{\phi_1,\dots,\phi_n\} = \operatorname{span}\{f_1,\dots,f_n\}$ and

$$\int_{-\infty}^{\infty}\psi_j(x)\phi_k(x)dx = \delta_{j,k} \qquad j,k=1,\dots,n.$$

In the case (1.1) we may write

$$K_n(x,y) = \sum_{j=0}^{n-1} P_j(x)Q_j(y)$$

where P_j is a monic polynomial of degree j for $j=0,\dots,n-1$, the dual functions $Q_0,\dots,Q_{n-1}$ are in the linear span of $f_1,\dots,f_n$, and the biorthogonality condition

$$\int_{-\infty}^{\infty} P_j(x)Q_k(x)dx = \delta_{j,k}, \qquad j,k=0,\dots,n-1 \tag{1.4}$$

is satisfied. In this case we can also consider the monic polynomial P_n of degree n such that (1.4) also holds for $j=n$. This polynomial is given by

$$P_n(x) = \mathbb{E}\left[\prod_{j=1}^n (x-x_j)\right]$$

where the averaging is over $(x_1, \ldots, x_n)$ in the polynomial ensemble (1.1). If the points $x_1, \ldots, x_n$ come from eigenvalues of a random matrix, then P_n is the average characteristic polynomial.

2. Known transformations

This paper discusses a number of transformations that preserve the structure of a polynomial ensemble. These transformations come from random matrix theory, and the typical setting is the following. We assume that X is a random matrix whose eigenvalues (or squared singular values) are distributed according to a polynomial ensemble (1.1). Then we perform a certain transformation to obtain from X a new random matrix Y, and the result is that the eigenvalues (or squared singular values) of Y are again a polynomial ensemble.

2.1. Product with Ginibre matrix. The first example of such a transformation comes from recent work of the author with Dries Stivigny [**21**]. It deals with the squared singular values of rectangular matrices. Recall that the squared singular values of a rectangular complex matrix X are the eigenvalues of X^*X. The transformation on X is multiplication by a complex Ginibre matrix, where a complex Ginibre matrix is a random matrix whose entries are independent standard complex Gaussians.

THEOREM 2.1. *Let n, l, ν be non-negative integers with $1 \leq n \leq l$. Let G be an $(n+\nu) \times l$ complex Ginibre matrix, and let X be a random matrix of size $l \times n$, independent of G, such that the squared singular values $x_1, \ldots, x_n$ are a polynomial ensemble* (1.1) *for certain functions $f_1, \ldots, f_n$ defined on $[0, \infty)$. Then the squared singular values $y_1, \ldots, y_n$ of $Y = GX$ are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j)\right]_{j,k=1}^n, \qquad \textit{all } y_j > 0, \tag{2.1}$$

where

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}, \qquad y > 0. \tag{2.2}$$

PROOF. See [**21**], where the proof is based on ideas taken from [**4, 5**]. □

Note that g_k in (2.2) is the Mellin convolution of $x \mapsto x^\nu e^{-x}$ with f_k.

Theorem 2.1 can be applied repeatedly and it follows that the multiplication with any number of complex Ginibre matrices preserves the structure of a polynomial ensemble for the squared singular values.

Theorem 2.1 was inspired by earlier results by Akemann et al. [**4, 5**] on products of random matrices. In these papers the authors considered products of complex Ginibre matrices (that is, X is also a complex Ginibre matrix) and they obtained the structure (2.1)–(2.2), where in this case the functions g_k in (2.1) are expressed as Meijer G-functions. This result has since then been used in [**22, 23**] to determine the large n scaling limit of the correlation kernel, and in [**2**] to calculate the Lyaponov exponents as the number of matrices in the product tends to infinity. See also [**15, 25, 26**] for other recent results on singular values of products of random matrices, and see [**3**] for a survey.

2.2. Product with a truncated unitary matrix. Theorem 2.1 has an extension to a product with a truncated unitary matrix. A $k \times l$ truncation T of a matrix U is the left upper submatrix of U of size $k \times l$. We assume that U is a Haar distributed random unitary matrix and then T is also a random matrix.

THEOREM 2.2. *Let n, m, l, ν be non-negative integers with $n \leq l \leq m$ and $m \geq n + \nu + 1$. Let T be an $(n+\nu) \times l$ truncation of a Haar distributed unitary matrix U of size $m \times m$. Let X be a random matrix of size $l \times n$, independent of U, such that the squared singular values $x_1, \ldots, x_n$ of X are a polynomial ensemble* (1.1) *for certain functions $f_1, \ldots, f_n$ defined on $[0, \infty)$. Then the squared singular values $y_1, \ldots, y_n$ of $Y = TX$ are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n}, \qquad \text{all } y_j > 0, \tag{2.3}$$

where

$$g_k(y) = \int_0^1 x^{\nu}(1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}, \qquad y > 0. \tag{2.4}$$

PROOF. See [**18**], and also Section 4 below. □

If we let $m \to \infty$ in Theorem 2.2, then $\sqrt{m}\,T$ tends in distribution to a complex Ginibre matrix. Also $(1 - \frac{x}{m})^{m-n-\nu-1}$ tends to e^{-x} as $m \to \infty$. In this way Theorem 2.1 can be obtained as a limiting case of Theorem 2.2.

Theorems 2.1 and 2.2 can be used repeatedly and it follows that the squared singular values of a product of any number of Ginibre matrices with any number of truncated unitary matrices are a polynomial ensemble.

2.3. Overview of the rest of the paper. Inspired by these results we give an overview of other transformations that preserve polynomial ensembles. The transformations are based on known random matrix theory calculations, see [**12, 17**], and our aim here is to emphasize the interpretation as a transformation of polynomial ensembles.

The first such transformation comes from matrix restrictions. Here we are working with a Hermitian matrix X and we remove one row and one column to obtain Y. If X is random with eigenvalues that are distributed as a polynomial ensemble then the eigenvalues of Y are also distributed as a polynomial ensemble. This is our first result, see Theorem 3.2. The proof relies on a fundamental result of Baryshnikov [**7**], see Theorem 3.1 below.

Then we extend this to the situation where X is a positive semidefinite matrix with a fixed number of zero eigenvalues. Again we find that matrix restriction for random matrices of this type leads to a transformation result for polynomial ensembles, see Theorem 4.2. Interestingly enough, we can make a connection with the product with a truncated unitary matrix, as we find in this way an alternative proof for Theorem 2.2.

In Section 5 we consider a transformation from X to $Y = X + vv^*$ where X is Hermitian, and v is a column vector of independent complex Gaussian entries. This rank-one modification is also a transformation of polynomial ensembles as we show in Proposition 5.1. The argument is based on a result of [**16**].

Finally, in Section 6 we consider a transformation where we extend the Hermitian matrix X by adding an extra column v with independent complex Gaussians, and an extra row $\begin{pmatrix} v^* & c \end{pmatrix}$ consisting of v^* and a real number c that has a real normal distribution. Under appropriate conditions on the variances, we again find a transformation of polynomial ensembles, see Proposition 6.2. This is based on [**1, 13**].

3. Matrix restrictions

Let X be an $n \times n$ Hermitian matrix with distinct eigenvalues $x_1 < x_2 < \cdots < x_n$. Let U be a Haar distributed unitary matrix of size $n \times n$ and let Y be the $(n-1)\times(n-1)$ principal submatrix of UXU^* with eigenvalues $y_1 \leq y_2 \leq \cdots \leq y_{n-1}$. With probability one we have strict interlacing of eigenvalues

$$x_1 < y_1 < x_2 < y_2 < \cdots < y_{n-1} < x_n. \tag{3.1}$$

The following theorem is due to Baryshnikov (reformulation of [**7**, Proposition 4.2]).

THEOREM 3.1. *If X and Y are as above, then the (random) eigenvalues $y_1, \dots, y_{n-1}$ of Y have the joint density*

$$(n-1)!\, \frac{\Delta_{n-1}(y)}{\Delta_n(x)} \tag{3.2}$$

on the subset of $\mathbb{R}^{n-1}$ defined by the inequalities (3.1).

The interlacing condition is expressed by the determinant

$$\det \left[\chi_{x_k \leq y_j}\right]_{j,k=1}^{n}, \qquad \chi_{x\leq y} = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

with $y_n := +\infty$. Indeed, for all mutually distinct values x_k and y_j, the determinant in (3.3) is 1 if and only if the interlacing condition holds and it is zero otherwise. The determinant in (3.3) has all ones in the last row. We can reduce it to an $(n-1)\times(n-1)$ determinant by subtracting the last column from every other column, and expanding along the last row. This results in the determinant $\det\left[\chi_{x_k \leq y_j < x_n}\right]_{j,k=1}^{n-1}$. It means that the density (3.2) can be written as

$$\frac{\Delta_{n-1}(y)}{\Delta_n(x)} \det \left[\chi_{x_k \leq y_j}\right]_{j,k=1}^{n} = \frac{\Delta_{n-1}(y)}{\Delta_n(x)} \det \left[\chi_{x_k \leq y_j < x_n}\right]_{j,k=1}^{n-1}, \\ y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}, \quad y_n := +\infty, \tag{3.4}$$

where (3.4) is considered as a probability density on the unordered eigenvalues of Y, i.e., as a probability density on $\mathbb{R}^{n-1}$. This accounts for the disappearance of the factor $(n-1)!$ from (3.2). Note that (3.4) is a polynomial ensemble with functions $y \mapsto \chi_{x_k \leq y < x_n}$, for $k = 1, 2, \dots, n-1$.

Let us now assume that X is random, independent of U, and that the eigenvalues of X are a polynomial ensemble. Then the eigenvalues of Y are again a polynomial ensemble. For this it is important that the normalization constant $\frac{1}{\Delta_n(x)}$ in (3.4) depends on X via the Vandermonde determinant $\Delta_n(x)$ in the denominator. We also need the Andreief identity, see [**11**, Chapter 3],

$$\int_{\mathcal{X}^n} \det\left[\phi_k(x_j)\right]_{j,k=1}^n \det\left[\psi_k(x_j)\right]_{j,k=1}^n d\mu(x_1)\cdots d\mu(x_n) \\ = n! \det\left[\int_{\mathcal{X}} \phi_j(x)\psi_k(x) d\mu(x)\right]_{j,k=1}^n \tag{3.5}$$

where μ is a measure on a space $\mathcal{X}$, $n \in \mathbb{N}$, and $\phi_1, \dots, \phi_n$, $\psi_1, \dots, \psi_n$ are arbitrary functions such that the integrals converge. The result is the following.

THEOREM 3.2. *Suppose that X is a random $n \times n$ Hermitian matrix whose eigenvalues are a polynomial ensemble* (1.1) *with certain functions $f_1, \dots, f_n$. Let Y be the principal submatrix of UXU^* of size $(n-1) \times (n-1)$, where U is a Haar distributed unitary matrix, independent of X. Then the eigenvalues $y_1, \dots, y_{n-1}$ of Y are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_{n-1}} \Delta_{n-1}(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n-1}, \tag{3.6}$$

with

$$g_k(y) = \int_{-\infty}^{y} \tilde{f}_k(x)\, dx, \qquad k = 1, \dots, n-1, \tag{3.7}$$

where $\tilde{f}_1, \dots, \tilde{f}_{n-1}$ are a basis for the vector space

$$\{f = \sum_{j=1}^{n} c_j f_j \mid c_1, \dots, c_n \in \mathbb{R}, \int_{-\infty}^{\infty} f(x)\, dx = 0\}. \tag{3.8}$$

PROOF. From (3.4) it follows after averaging over the polynomial ensemble (1.1) that the eigenvalues of Y have joint density

$$\frac{1}{n! Z_n} \Delta_{n-1}(y) \int_{\mathbb{R}^n} \det \left[\chi_{x_k \leq y_j}\right]_{j,k=1}^{n} \det \left[f_k(x_j)\right]_{j,k=1}^{n} dx_1 \cdots dx_n$$

with $y_n := +\infty$. Because of (3.5) we find that this is

$$\frac{1}{Z_n} \Delta_{n-1}(y) \det \left[\int_{-\infty}^{y_j} f_k(x) dx\right]_{j,k=1}^{n}. \tag{3.9}$$

Since $y_n = +\infty$, the last row of the second determinant in (3.9) contains the constants $\int_{-\infty}^{\infty} f_k(x) dx$. Change from $f_1, \dots, f_n$ to another basis $\tilde{f}_1, \dots \tilde{f}_n$ where $\tilde{f}_1, \dots, \tilde{f}_{n-1}$ belong to the subspace (3.8). Then after performing suitable column operations we obtain the density

$$\frac{1}{\tilde{Z}_{n-1}} \Delta_{n-1}(y) \det \left[\int_{-\infty}^{y_j} \tilde{f}_k(x) dx\right]_{j,k=1}^{n}.$$

Then $\int_{-\infty}^{\infty} \tilde{f}_n(x) dx \neq 0$, since otherwise the full last row in the determinant would be zero. By expanding the determinant along the last row we obtain (3.6) with functions (3.7) and a possibly different constant $\tilde{Z}_{n-1}$. $\square$

An analogous result holds for singular values.

COROLLARY 3.3. *Suppose that X is a random $(n+\nu) \times n$ matrix whose squared singular values are a polynomial ensemble* (1.1). *Let Y be the $(n+\nu) \times (n-1)$ left upper submatrix of XU where U is a Haar distributed unitary matrix, independent of X. Then the squared singular values $y_1, \dots, y_{n-1}$ of Y are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_{n-1}} \Delta_{n-1}(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n-1}, \qquad \text{all } y_j > 0, \tag{3.10}$$

with

$$g_k(y) = \int_{0}^{y} \tilde{f}_k(x) dx, \qquad k = 1, \dots, n-1 \tag{3.11}$$

where $\tilde{f}_1, \dots, \tilde{f}_{n-1}$ *are a basis for the vector space*

$$\{f = \sum_{j=1}^{n} c_j f_j \mid c_1, \dots, c_n \in \mathbb{R}, \int_0^\infty f(x)dx = 0\}. \tag{3.12}$$

PROOF. We can apply Theorem 3.2 since Y^*Y is the principal submatrix of size $(n-1) \times (n-1)$ of U^*X^*XU. The integration in (3.11) and (3.12) starts at 0 since the functions are defined for $x \geq 0$ only. □

4. Restrictions of positive semidefinite matrices

The following is a variation on Theorem 3.1. It can also be obtained as a special case of [**17**, Corollary 1].

PROPOSITION 4.1. *Let* $m \geq n+1$ *and let* X *be an* $m \times m$ *positive semidefinite Hermitian matrix with* n *simple non-zero eigenvalues* $0 < x_1 < x_2 < \dots < x_n$ *and an eigenvalue* 0 *of multiplicity* $m - n \geq 1$. *Let* Y *be the* $(m-1) \times (m-1)$ *principal submatrix of* UXU^* *where* U *is a Haar distributed unitary matrix of size* $m \times m$. *Then with probability one,* Y *has exactly* n *non-zero eigenvalues* $0 < y_1 < y_2 < \dots < y_n$ *that satisfy the inequalities*

$$0 < y_1 < x_1 < y_2 < x_2 < \dots < x_{n-1} < y_n < x_n, \tag{4.1}$$

and these non-zero eigenvalues have the joint density

$$\frac{(m-1)!}{(m-n-1)!} \left(\prod_{k=1}^{n} \frac{y_k^{m-n-1}}{x_k^{m-n}} \right) \frac{\Delta_n(y)}{\Delta_n(x)} \tag{4.2}$$

restricted to the subset of $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ *defined by the inequalities* (4.1).

PROOF. For $m = n+1$ this follows immediately from Theorem 3.1 and so we assume in the proof that $m \geq n+2$. We approximate X by a matrix A with eigenvalues $a_1 < \dots < a_{m-n} < x_1 < \dots < x_n$ with a_j's close to zero. Let B be the principal submatrix of UAU^* of size $(m-1) \times (m-1)$, which with probability one has distinct eigenvalues $b_1 < \dots < b_{m-n-1} < y_1 < \dots < y_n$ that interlace with the eigenvalues of A. By Theorem 3.1 the joint density of these eigenvalues is

$$(m-1)! \frac{\Delta_{m-n-1}(b) \prod_{j,k}(y_k - b_j)}{\Delta_{m-n}(a) \prod_{j,k}(x_k - a_j)} \frac{\Delta_n(y)}{\Delta_n(x)} \tag{4.3}$$

on the subset of $\mathbb{R}^m$ given by the interlacing relations. The induced density on $y_1, \dots, y_n$ is

$$(m-1)! \left(\int_{a_1}^{a_2} db_1 \cdots \int_{a_{m-n-1}}^{a_{m-n}} db_{m-n-1} \frac{\Delta_{m-n-1}(b) \prod_{j,k}(y_k - b_j)}{\Delta_{m-n}(a) \prod_{j,k}(x_k - a_j)} \right) \frac{\Delta_n(y)}{\Delta_n(x)}. \tag{4.4}$$

In the limit where all $a_j \to 0$, $j = 1, \dots, m-n$, we also have $b_j \to 0$, $j = 1, \dots, m-n-1$. Then the factors $\prod_{j,k}(y_k - b_j)$ and $\prod_{j,k}(x_k - a_j)$ in (4.4) tend to $\prod_k y_k^{m-n-1}$ and $\prod_k x_k^{m-n}$, respectively. The remaining $m-n-1$ fold integral can be evaluated as

$$\int_{a_1}^{a_2} db_1 \dots \int_{a_{m-n-1}}^{a_{m-n}} db_{m-n-1} \frac{\Delta_{m-n-1}(b)}{\Delta_{m-n}(a)} = \frac{1}{(m-n-1)!}, \tag{4.5}$$

since $\Delta_{m-n-1}(b) = \det \left[b_j^{k-1}\right]_{j,k=1}^{m-n-1}$ and

$$\int_{a_1}^{a_2} db_1 \dots \int_{a_{m-n-1}}^{a_{m-n}} db_{m-n-1} \Delta_{m-n-1}(b)$$
$$= \det \left[\int_{a_j}^{a_{j+1}} b^{k-1} db \right]_{j,k=1}^{m-n-1} = \frac{1}{(m-n-1)!} \det \left[a_{j+1}^k - a_j^k\right]_{j,k=1}^{m-n-1}$$

which leads to (4.5) since the last determinant is easily seen to be $\Delta_{m-n}(a)$. Note that the right-hand side (4.5) does not depend on $a_1, \dots a_{m-n}$. The result is the joint density (4.2) for the non-zero eigenvalues of Y. $\square$

The inequalities (4.1) are encoded by the determinant

$$\det \left[\chi_{0<y_j<x_k}\right]_{j,k=1}^n$$

which for strictly increasing $y_1 < y_2 < \cdots < y_n$ is 1 if the interlacing (4.1) holds and 0 otherwise. Then (4.2) can be alternatively written as

$$\frac{(m-1)!}{n!(m-n-1)!} \left(\prod_{k=1}^n \frac{y_k^{m-n-1}}{x_k^{m-n}} \right) \frac{\Delta_n(y)}{\Delta_n(x)} \det \left[\chi_{0<y_j<x_k}\right]_{j,k=1}^n \tag{4.6}$$

which is now considered as a density on $[0,\infty)^n$ for unordered eigenvalues. Note that (4.6) is a polynomial ensemble on $[0,\infty)$ with functions $y \mapsto y^{m-n-1}\chi_{0<y<x_k}$ for $k = 1, \dots, n$.

Theorem 4.2. *Let $n \leq m-1$ and $\nu \leq m-n-1$ be positive integers. Let X be a random positive semidefinite Hermitian matrix of size $m \times m$ with a zero eigenvalue of multiplicity $m - n \geq 1$ and non-zero eigenvalues $x_1, \dots, x_n$ that are a polynomial ensemble* (1.1) *for certain functions $f_1, \dots, f_n$ on $[0,\infty)$. Let Y be the principal submatrix of UXU^* of size $(n+\nu) \times (n+\nu)$, where U is a Haar distributed unitary matrix, independent of X. Then, with probability one, Y has exactly n non-zero eigenvalues $y_1, \dots, y_n$, and these non-zero eigenvalues are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j)\right]_{j,k=1}^n \qquad \textit{all } y_j > 0, \tag{4.7}$$

where

$$g_k(y) = \int_0^1 x^\nu (1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}. \tag{4.8}$$

Proof. We first assume that $\nu = m - n - 1$. Then Y is obtained from UXU^* by removing one row and column and we can apply Proposition 4.1 and in particular its reformulation in (4.6). Averaging (4.6) over the polynomial ensemble (1.1) we obtain the joint density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \int_{[0,\infty)^n} \left(\prod_{k=1}^n \frac{y_k^{m-n-1}}{x_k^{m-n}} \right) \det \left[f_k(x_j)\right]_{j,k=1}^n \det \left[\chi_{0<y_j<x_k}\right]_{j,k=1}^n dx_1 \cdots dx_n,$$

for a certain constant $\tilde{Z}_n$ (which also depends on m). By the Andreief identity (3.5) this leads to

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[\int_0^\infty \frac{y_j^{m-n-1}}{x^{m-n}} f_k(x) \chi_{0<y_j<x} \, dx \right]_{j,k=1}^n$$

with a new constant $\tilde{Z}_n$. This is a polynomial ensemble (4.7) with functions

$$\begin{aligned} g_k(y) &= \int_0^\infty \frac{y^{m-n-1}}{x^{m-n}} f_k(x) \chi_{0<y<x}\, dx \\ &= \int_y^\infty \frac{y^{m-n-1}}{x^{m-n}} f_k(x)\, dx, \qquad y > 0. \end{aligned} \tag{4.9}$$

The substitution $x \mapsto \frac{y}{x}$ in (4.9) leads to the expression (4.8) with $\nu = m - n - 1$. This is the Mellin convolution of f_k with the function χ_{m-n-1} where we define

$$\chi_k : [0, \infty) \to \mathbb{R} : x \mapsto x^k \chi_{0<x<1}.$$

Thus $g_k = \chi_{m-n-1} * f_k$, if $\nu = m - n - 1$, where $*$ is used here for the Mellin convolution

$$(f * g)(y) = \int_0^\infty f(x) g\left(\frac{y}{x}\right) \frac{dx}{x}.$$

For general $\nu \leq m - n - 1$ we can use the above argument repeatedly, and we find a polynomial ensemble (4.7) with functions g_k that are iterated Mellin convolutions of the functions f_k, namely

$$g_k = \chi_\nu * \chi_{\nu+1} * \cdots * \chi_{m-n-1} * f_k, \qquad k = 1, \ldots, n.$$

It is easy to calculate that

$$(\chi_\nu * \chi_{\nu+1} * \cdots * \chi_{m-n-1})(x) = x^\nu (1-x)^{m-n-\nu-1}, \qquad 0 < x < 1,$$

and thus we obtain the formula (4.8) for the functions g_k. □

An attentive reader may have noticed that the formula for g_k in (4.8) coincides with the one appearing in (2.4) in Theorem 2.2. This is no coincidence since we can use Theorem 4.2 to give an alternative proof of Theorem 2.2, which was first proved in [**18**].

Proof of Theorem 2.2. Let X be an $l \times n$ matrix, and put

$$\tilde{X} = \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix}$$

which is an $m \times m$ matrix with $m - l$ rows and columns containing only zeros. It is clear that the squared singular values of X are equal to non-zero eigenvalues of $\tilde{X}$.

Also if U is a unitary matrix of size $m \times m$ and T is its left upper block of size $(n + \nu) \times l$ then

$$U\tilde{X}U^* = \begin{pmatrix} TXX^*T^* & * \\ * & * \end{pmatrix}$$

where $*$ denotes a certain unspecified entry, whose value is not important for us. In other words, TXX^*T^* is equal to the principal submatrix of $U\tilde{X}U^*$ of size $(n + \nu) \times (n + \nu)$. Assuming U is Haar distributed over the unitary group, and the squared singular values of X are a polynomial ensemble (1.1), independent of U, we then find from Theorem 4.2 that the non-zero eigenvalues of TXX^*T^* are a polynomial ensemble (2.1) with functions (4.8). The non-zero eigenvalues of TXX^*T^* are the same as the squared singular values of TX and Theorem 2.2 follows. □

5. Rank one modification

Let X be a Hermitian $n \times n$ matrix with eigenvalues $x_1 < x_2 < \cdots < x_n$. We take $Y = X + vv^*$ where v is a vector of length n. Then the eigenvalues y_j of Y interlace with those of X, as follows from the Courant-Fischer Theorem, see e.g. [**24**, chapter 7.5]. We let $v = (v_1, \ldots, v_n)^t$ be a vector of independent complex random variables whose real and imaginary parts are independent and have a $N(0, 1/2)$ distribution. Then the distribution of the eigenvalues of Y is given in [**16**, Appendix E] as

$$\left(\prod_{j=1}^{n} e^{-(y_j - x_j)}\right) \frac{\Delta_n(y)}{\Delta_n(x)} \tag{5.1}$$

on the subset of $\mathbb{R}^n$ given by the interlacing conditions

$$x_1 < y_1 < x_2 < \cdots < x_n < y_n. \tag{5.2}$$

The following result is an immediate consequence.

Proposition 5.1. *Let* $\operatorname{Re} v_j$, $\operatorname{Im} v_j$, *for* $j = 1, \ldots, n$ *be mutually independent normal random variables with mean zero and variance* $1/2$. *Let* X *be a random Hermitian matrix of size* $n \times n$, *independent of* $v = (v_1, \ldots, v_n)^t$, *whose eigenvalues are a polynomial ensemble* (1.1) *with certain functions* $f_1, \ldots, f_n$. *Then the eigenvalues* $y_1, \ldots, y_n$ *of* $Y = X + vv^*$ *are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n}, \tag{5.3}$$

where

$$g_k(y) = \int_0^\infty e^{-x} f_k(y - x)\, dx, \qquad k = 1, \ldots, n. \tag{5.4}$$

Thus g_k is the convolution of f_k with $x \mapsto e^{-x} \chi_{x \geq 0}$.

Proof. The interlacing (5.2) is encoded by a determinant, and it follows that (5.1) is a polynomial ensemble

$$\frac{1}{n!} \left(\prod_{j=1}^{n} e^{-(y_j - x_j)}\right) \frac{\Delta_n(y)}{\Delta_n(x)} \det \left[\chi_{x_k < y_j}\right]_{j,k=1}^{n}, \tag{5.5}$$

where now we disregard the ordering of the y_j's and consider (5.5) as a probability density on $\mathbb{R}^n$.

We average over $x_1, \ldots, x_n$ distributed as in (1.1). By Andreief's identity (3.5), we obtain for the density of the eigenvalues of Y

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[\int_{-\infty}^{y_j} e^{-(y_j - x)} f_k(x)\, dx\right]_{j,k=1}^{n}.$$

Changing variables $x \mapsto y_j - x$ in the integral in the determinant, we arrive at (5.3) with functions (5.4). $\square$

Here is a variation on the same theme.

PROPOSITION 5.2. *Let $n, \nu \geq 1$. Let $\operatorname{Re} v_j$, $\operatorname{Im} v_j$, for $j = 1, \ldots, n+\nu$ be mutually independent normal random variables with mean zero and variance $1/2$. Let X be a random $(n+\nu) \times (n+\nu)$ positive semidefinite Hermitian matrix, independent of $v_1, \ldots, v_n$, with exactly n positive eigenvalues $x_1, \ldots, x_n$ that are a polynomial ensemble* (1.1) *with certain functions $f_1, \ldots, f_n$ on $[0, \infty)$. Then, almost surely, $Y = X + vv^*$ has an eigenvalue zero of multiplicity $\nu - 1$ and $n+1$ positive eigenvalues $y_1, \ldots, y_{n+1}$ that are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_{n+1}} \Delta_{n+1}(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n+1}, \qquad \text{all } y_j > 0, \tag{5.6}$$

with functions

$$\begin{aligned} g_1(y) &= y^{\nu-1} e^{-y}, \\ g_{k+1}(y) &= y^{\nu-1} e^{-y} \int_c^y x^{-\nu} e^x f_k(x) dx, \qquad k = 1, \ldots, n, \end{aligned} \tag{5.7}$$

for $y > 0$, where $c \in (0, \infty)$ is an arbitrary but fixed positive real number.

We may also take $c = 0$ or $c = \infty$ in (5.7) provided that the integrals are all convergent.

PROOF. We approximate X by A with distinct eigenvalues $a_1 < \cdots < a_\nu < x_1 < \cdots x_n$ where the a_j are close to 0. Then $B = A + vv^*$ has eigenvalues $b_1 < \cdots < b_{\nu-1} < y_1 < \cdots < y_{n+1}$ that interlace with those of A, with a joint density, see (5.1),

$$\prod_{j=1}^{\nu-1} e^{-b_j} \prod_{j=1}^{n+1} e^{-y_j} \prod_{j=1}^{\nu} e^{a_j} \prod_{j=1}^{n} e^{x_j} \frac{\Delta_{\nu-1}(b) \left(\prod_{j=1}^{\nu-1} \prod_{k=1}^{n+1} (y_k - b_j)\right) \Delta_{n+1}(y)}{\Delta_\nu(a) \left(\prod_{j=1}^{\nu} \prod_{k=1}^{n} (x_k - a_j)\right) \Delta_n(x)}$$

subject to the interlacing conditions.

We restrict this to the y-variables by integrating out $b_1, \ldots, b_{\nu-1}$. This gives the joint density for $y_1, \ldots, y_{n+1}$

$$\begin{aligned} &\prod_{j=1}^{n+1} e^{-y_j} \prod_{j=1}^{n} e^{x_j} \prod_{j=1}^{\nu} e^{a_j} \frac{\Delta_{n+1}(y)}{\Delta_\nu(a) \left(\prod_{j=1}^{\nu} \prod_{k=1}^{n} (x_k - a_j)\right) \Delta_n(x)} \\ &\qquad \int_{a_1}^{a_2} db_1 \cdots \int_{a_{\nu-1}}^{a_\nu} db_{\nu-1} \prod_{j=1}^{\nu-1} e^{-b_j} \Delta_{\nu-1}(b) \left(\prod_{j=1}^{\nu-1} \prod_{k=1}^{n+1} (y_k - b_j)\right). \end{aligned}$$

In the limit where all $a_j \to 0$ we also have that all $b_j \to 0$ because of the interlacing. Then $A \to X$, $B \to Y$, and using also (4.5) we find the limiting joint density for the nonzero eigenvalues $y_1, \ldots, y_{n+1}$ of Y

$$\frac{1}{Z_{n+1}} \left(\prod_{j=1}^{n+1} y_j^{\nu-1} e^{-y_j}\right) \left(\prod_{j=1}^{n} x_j^{-\nu} e^{x_j}\right) \frac{\Delta_{n+1}(y)}{\Delta_n(x)}$$

subject to the interlacing $0 < y_1 < x_1 < \cdots < x_n < y_{n+1}$. The interlacing is encoded by the determinant $\det \left[\chi_{y_j < x_k < y_{n+1}}\right]_{j,k=1}^{n}$.

Next, averaging over the polynomial ensemble (1.1) and using the Andreief identity (3.5), we find in a now familiar fashion the joint density

$$\frac{1}{\tilde{Z}_{n+1}} \Delta_{n+1}(y) \det \left[y_j^{\nu-1} e^{-y_j} \int_0^\infty x^{-\nu} e^x f_k(x) \chi_{y_j < x < y_{n+1}} dx \right]_{j,k=1}^n \tag{5.8}$$

for the eigenvalues of Y. The $n \times n$ determinant in (5.8) is extended to an $(n+1) \times (n+1)$ determinant by adding first a row with zeros and then a column with ones. Then after elementary column operations we easily arrive at the polynomial ensemble (5.6) with functions (5.7) and a possibly different constant $\tilde{Z}_{n+1}$. □

The two Propositions 5.1 and 5.2 have the following consequences regarding squared singular values of an extension of a matrix by one row or one column.

COROLLARY 5.3. *Suppose $\nu \geq 0$. Let X be an $(n+\nu) \times n$ random matrix with squared singular values $0 < x_1 < \cdots < x_n$ that form a polynomial ensemble* (1.1) *with certain functions $f_1, \ldots, f_n$ on $[0, \infty)$. Let $Y = \begin{pmatrix} X \\ v^* \end{pmatrix}$ with v a random vector of independent complex Gaussians as in Proposition 5.1, which is independent of X. Then the squared singular values $y_1, \ldots, y_n$ of Y are a polynomial ensemble* (5.3) *with functions*

$$g_k(y) = \int_0^y e^{-x} f_k(y-x)\, dx. \tag{5.9}$$

PROOF. The squared singular values of X are the eigenvalues of X^*X. The squared singular values of Y are the eigenvalues of $\begin{pmatrix} X^* & v \end{pmatrix} \begin{pmatrix} X \\ v^* \end{pmatrix} = X^*X + vv^*$. Thus the result follows from Proposition 5.1. The integration in (5.9) extends to y only, and not to ∞ as in (5.4), since $f_k(x)$ is defined for $x \geq 0$ only, and we consider $f_k(y-x)$ to be zero if $x > y$.

□

COROLLARY 5.4. *Suppose $\nu \geq 1$. Let X be an $(n+\nu) \times n$ matrix with squared singular values $0 < x_1 < \cdots < x_n$ that are a polynomial ensemble* (1.1) *with certain functions $f_1, \ldots, f_n$ on $[0, \infty)$. Let $Y = \begin{pmatrix} X & v \end{pmatrix}$ with v a random vector of independent complex Gaussians as in Proposition 5.2, which is independent of X. Then the squared singular values $y_1, \ldots, y_{n+1}$ of Y are a polynomial ensemble* (5.6) *with functions* (5.9).

PROOF. The squared singular values of X are the non-zero eigenvalues of XX^*, and the squared singular values of Y are the non-zero eigenvalues of $YY^* = XX^* + vv^*$. Thus the result follows from Proposition 5.2. □

It is interesting to note that a combination of Corollaries 5.3 and 5.4 leads to the proof of one of the classical results of random matrix theory [**6, 12**], namely that the squared singular values of a complex Ginibre matrix are distributed as a Laguerre ensemble. See also [**12**, Chapter 4.3.3] for a similar approach, and [**13**] for related results.

COROLLARY 5.5. *Suppose X is an $m \times n$ random matrix such that $\operatorname{Re} X_{i,j}$, $\operatorname{Im} X_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are independent normal random variables with* *mean zero and variance* $1/2$. *Suppose* $\nu = m - n \geq 0$. *Then the squared singular values* $x_1, \dots, x_n$ *of* X *have the joint density*

$$\frac{1}{Z_n} \prod_{i<j} (x_j - x_i)^2 \prod_{j=1}^{n} x_j^{\nu} e^{-x_j}, \qquad \text{all } x_j > 0. \tag{5.10}$$

Proof. We use induction. It is easy to check Corollary 5.5 for $m = n = 1$. Assume Corollary 5.5 holds for certain $m, n \geq 1$. Note that (5.10) is a polynomial ensemble with functions $f_k(x) = x^{\nu+k-1} e^{-x}$ for $k = 1, \dots, n$. Then by Corollary 5.3 it will follow that Corollary 5.5 also holds for $m+1$ and n, and by Corollary 5.4 it holds for m and $n+1$, provided that $m > n$. The calculations are straightforward and we do not give them explictly here. □

6. Matrix extensions

In this final section we start from an $n \times n$ Hermitian matrix X and we are going to extend it to an $(n+1) \times (n+1)$ Hermitian matrix by adding one row and one column. We write

$$Y = \begin{pmatrix} X & v \\ v^* & c \end{pmatrix} \tag{6.1}$$

where $v = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^t$ is a complex column vector, and $c \in \mathbb{R}$ is real.

The following result was given by Forrester [**13**] and Adler, Van Moerbeke and Wang [**1**], see also [**12**, Chapter 4.3.2] and [**16**, section 3.1], where the focus is on the situation where X is an $n \times n$ GUE matrix.

Theorem 6.1. *Suppose* c, $\operatorname{Re} v_j$, *and* $\operatorname{Im} v_j$ *for* $j = 1, \dots, n$ *are independent normal random variables with mean zero, where* c *has variance* 1 *and* $\operatorname{Re} v_j$, $\operatorname{Im} v_j$ *have variance* $1/2$. *Assume* X *has simple eigenvalues* $x_1 < x_2 < \cdots < x_n$. *Then with probability one, the ordered eigenvalues* $y_1 \leq y_2 \leq \cdots \leq y_{n+1}$ *of* Y *are simple, and strictly interlace with those of* X*:*

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < y_{n+1}. \tag{6.2}$$

In addition, the eigenvalues of Y *have the probability density*

$$\frac{1}{\sqrt{2\pi}} \left(\prod_{j=1}^{n+1} e^{-\frac{1}{2} y_j^2} \right) \left(\prod_{j=1}^{n} e^{\frac{1}{2} x_j^2} \right) \frac{\Delta_{n+1}(y)}{\Delta_n(x)} \tag{6.3}$$

on the subset of $\mathbb{R}^{n+1}$ *defined by the interlacing condition* (6.2).

Proof. See Lemma 1 in [**1**] or Proposition 6 in [**13**]. □

As before, there is an immediate consequence of Theorem 6.1 to polynomial ensembles.

Proposition 6.2. *Suppose* c, $\operatorname{Re} v_j$, *and* $\operatorname{Im} v_j$ *for* $j = 1, \dots, n$ *are mutually independent normal random variables with mean zero, where* c *has variance* 1 *and* $\operatorname{Re} v_j$, $\operatorname{Im} v_j$ *have variance* $1/2$. *Suppose that* X *is a random Hermitian matrix of size* $n \times n$, *independent of* c *and* v, *whose eigenvalues are a polynomial ensemble* (1.1) *with certain functions* $f_1, \dots, f_n$. *Then the eigenvalues* $y_1, \dots, y_{n+1}$ *of* Y *given by* (6.1) *are a polynomial ensemble*

$$\frac{1}{\tilde{Z}_{n+1}} \Delta_{n+1}(y) \det \left[g_k(y_j) \right]_{j,k=1}^{n+1}, \tag{6.4}$$

with functions

$$\begin{aligned} g_1(y) &= e^{-\frac{1}{2}y^2}, \\ g_{k+1}(y) &= e^{-\frac{1}{2}y^2} \int_0^y e^{\frac{1}{2}x^2} f_k(x)\,dx, \qquad k = 1, \ldots, n. \end{aligned} \tag{6.5}$$

PROOF. The eigenvalues of X are distinct with probability one. We order them, say $x_1 < x_2 < \cdots < x_n$.

We use an interlacing determinant as in (3.3) to write the density (6.3) as

$$\frac{1}{Z_{n+1}} \left(\prod_{j=1}^{n+1} e^{-\frac{1}{2}y_j^2} \right) \left(\prod_{j=1}^{n} e^{\frac{1}{2}x_j^2} \right) \frac{\Delta_{n+1}(y)}{\Delta_n(x)} \det \left[\chi_{y_j \leq x_k} \right]_{j,k=1}^{n+1}$$

with $x_{n+1} = +\infty$ and a certain constant Z_{n+1}, which is also

$$\frac{1}{Z_{n+1}} \left(\prod_{j=1}^{n+1} e^{-\frac{1}{2}y_j^2} \right) \left(\prod_{j=1}^{n} e^{\frac{1}{2}x_j^2} \right) \frac{\Delta_{n+1}(y)}{\Delta_n(x)} \det \left[\chi_{y_j \leq x_k < y_{n+1}} \right]_{j,k=1}^{n}. \tag{6.6}$$

Then averaging (6.6) with respect to the polynomial ensemble (1.1) and using the Andreief identity (3.5) we obtain for the density of $y_1, \ldots, y_{n+1}$,

$$\frac{1}{\tilde{Z}_{n+1}} \left(\prod_{j=1}^{n+1} e^{-\frac{1}{2}y_j^2} \right) \Delta_{n+1}(y) \det \left[\int_{y_j}^{y_{n+1}} e^{\frac{1}{2}x^2} f_k(x) dx \right]_{j,k=1}^{n}.$$

We extend the last determinant to an $(n+1) \times (n+1)$ determinant by adding first a row of length n with entries $\int_0^{y_{n+1}} e^{\frac{1}{2}x^2} f_k(x) dx$ for $k = 1, \ldots, n$ and then a column $\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^t$ of length $n+1$. We subtract the last row from each of the other rows and change the sign in each of the entries in rows 1 up to n. We cyclically permute the columns in order to bring the last column to the beginning. Then we introduce the factor $e^{-\frac{1}{2}y_j^2}$ into the jth row of the determinant, and obtain the result (6.4) with functions (6.5). □

As a special case, we consider the polynomial ensemble (1.1) with functions $f_k(x) = x^{k-1} e^{-\frac{1}{2}x^2}$ for $k = 1, \ldots, n$. This is the same as

$$\frac{1}{Z_n} \Delta_n(x)^2 \prod_{j=1}^{n} e^{-\frac{1}{2}x_j^2}. \tag{6.7}$$

Then by (6.5) we get $g_1(y) = e^{-\frac{1}{2}y^2}$ and

$$g_{k+1}(y) = e^{-\frac{1}{2}y^2} \int_0^y x^{k-1} dx = \frac{1}{k} y^k e^{-\frac{1}{2}y^2} \qquad \text{for } k = 1, \ldots, n.$$

The prefactor $\frac{1}{k}$ is immaterial and it follows from Proposition 6.2 that the density function for the eigenvalues of Y is

$$\frac{1}{Z_{n+1}} \Delta_{n+1}(y)^2 \prod_{j=1}^{n+1} e^{-\frac{1}{2}y_j^2}.$$

It is well-known that (6.7) is the density of eigenvalues of GUE random matrix [**6, 12**] and we conclude, as already noted in [**12**, Chapter 4.3.2], that we can use Proposition 6.2 to give an inductive proof of this basic result of random matrix theory.

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